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Income-Related Health Transfers Principles and Orderings of Joint Distributions of Income and Health

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Abstract

The objective of this article is to provide the analyst with the necessary tools that allow for a robust ordering of joint distributions of health and income. We contribute to the literature on the measurement and inference of socioeconomic health inequality in three distinct but complementary ways. First, we provide a formalization of the socioeconomic health inequality-specific ethical principle introduced by Erreygers Clark and van Ourti, (2012). Second, we propose new graphical tools and dominance tests for the identification of robust orderings of joint distributions of income and health associated with this new ethical principle. Finally, based on both pro-poor and proextreme ranks ethical principles we address a very important aspect of dominance literature: the inference. To illustrate the empirical relevance of the proposed approach, we compare joint distributions of income and a health-related behaviour in the United States in 1997 and 2014.

JEL Classification: D63; I10

Keywords: health concentration curves; health range curves; socioeconomic health inequality; dominance; inference

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1 Introduction

While some may consider that income inequalities are the result of differential effort and skill, there are domains in which inequality is perceived (by many) as a social injustice; health inequality is one of them (Tobin, 1970). This is why measuring socioeconomic health inequality is important from a social perspective and is critical when evaluating the impact of health policy reforms on the distribution of population's health. There is a large body of literature on the measurement of socioeconomic health inequalities most of which has focused on the properties and issues arising from the use of these indices as well as the ethical principles they should obey. Some argue that the analyst should be concerned with inequalities that occur in the lower part of the distribution of socioeconomic status (Wagstaff, 2002) and others maintain that the analyst should be more concerned with deviations occurring away from the median of the socioeconomic status (Erreygers, Clarke and Van Ourti, 2012). While the desirable ethical principles for these measures may still be on the debate table, this paper adopts a unified approach by including both possibilities. As such, the overarching objective of this paper is to provide the analyst with the necessary tools that allow for a robust ordering of joint distributions of health and income including the associated statistical inference compatible with both ethical principles.

This paper contributes to the literature on socioeconomic health inequality measurement and inference in three distinct but complementary ways. First, it contributes to the literature that formalizes the ethical principles underlying socioeconomic health inequality indices by offering a formalization of Erreygers, Clarke and Van Ourti's (2012) view of what is considered a desirable property for these indices. In doing so, it provides a formal presentation of the ethical principles associated with indices that pass the *upside-down* test and coin these principles as the *symmetry around the median* principle (at the second order) and the *pro-extreme rank* principles (at higher orders). Second, it contributes to the socioeconomic health inequality measurement literature by introducing new graphical tools associated with these principles, a new class of range curves, and by deriving the associated dominance conditions. These range curves have a role analogous to the one played by health concentration curves where the analyst adopts *pro-poor* ethical principles. Developing new dominance conditions (for these new range curves) will help the analyst identify robust orderings when operating under the assumption that the *symmetry around the median* principle and *pro-extreme rank* principles are desirable. Finally, it contributes to the literature on hypothesis testing for dominance conditions by providing estimators of health concentration curves and health range curves as well as consistent testing methods for dominance compatible with both ethical principles.

The remaining of this paper is organized as follows. In section 2, we provide a brief review of the literature on the measures of socioeconomic health inequality, the basic ethical principles on which they are founded as well as the associated literature on inference. In section 3, we describe the measurement framework in which we are operating and discuss the associated basic ethical principles. In section 4, we examine higher order ethical principles for *pro-poor* and *pro-extreme rank* ethical principles then define the sets off indices obeying these principles. In section 5, we present the health concentration curve, the s-health concentration curves, the health range curve, the s-health range curves and their respective generalized versions. We also develop dominance conditions to identify robust orderings for the sets of indices developed in section 4. In section 6, we present the natural estimators for the curves presented in section 5 and develop the statistical inference to test for dominance. Finally, in section 7, we provide an empirical illustration using information on cigarette consumption and overweightedness from National Health Interview Survey (NHIS) in 1997 and 2014.

2 Literature Review

This paper is related to two main strains in the literature, the literature on the measurement of socioeconomic health inequality and the literature on hypothesis testing for dominance in the context of inequality.

The most traditional measure of socioeconomic health inequality is the concentration index proposed by Wagstaff, Paci and van Doorslaer (1991). It has a mathematical structure that assumes a very specific form and level of aversion to socioeconomic health inequality. Wagstaff (2002) argues that it may be desirable to consider other levels of inequality aversion than the one that is implicitly imposed in the standard concentration index. He suggests a parametric class of indices: the extended health concentration indices. This proposed class of indices incorporates a parameter that allows for a wider range of levels for aversion to socioeconomic health inequality than the one embodied in the principle of income related health transfers.

Erreygers, Clarke and Van Ourti (2012) highlight that the use of extended health concentration indices imposes a specific ethical view on what constitutes an increase in aversion to socioeconomic health inequality; they label it *pro-poor* transfer sensitivity ethical position. In the context of *pro-poor* transfer sensitivity, increasing aversion to socioeconomic health inequality is achieved by increasing the weight of transfers occurring at lower ranks of socioeconomic distribution. This widely used ethical position is based on a concept developed in the (unidimensional) income inequality literature and is adapted to fit the (bi-dimensional) socioeconomic health inequality context. Erreygers, Clarke and Van Ourti (2012) consider that *pro-poor* transfer sensitivity is debatable as the analyst may want to consider other ethical principles. They also argue that this ethical principle is not appropriate in a bi-dimensional context such as socioeconomic health inequality and propose new ethical principles that we will label in this paper as the *symmetry around the* *median* principle and *pro-extreme rank* transfer sensitivity principles. Based on this principle, an increase in socioeconomic health inequality is achieved by increasing the weights on transfers occurring further away from the median of socioeconomic statuses. Makdissi and Yazbeck (2014) formalize the definition of *pro-poor* transfer sensitivity principles and introduce higher orders of health concentration curves, the s-health concentration curves. They show how these curves can be used to identify robust orderings of health distributions for indices obeying *pro-poor* transfer sensitivity principles. From this perspective, this paper is related to Makdissi and Yazbeck (2014) yet is different from it in two distinct ways. First, it formalizes the *pro-extreme rank* principles introduced by Erreygers, Clarke and Van Ourti (2012) and derives the corresponding higher order ethical principles. Second, it proposes new graphical tools associated with these principles; the health range curve and the s-health range curves. These curves will be used to derive necessary and sufficient conditions for robust orderings of joint distributions of income and health.

Compared to the literature on socioeconomic health inequality measures, the literature on the statistical inference for these measures is scant as most it focused on income inequality measures (Kakwani, Wagstaff and Van Doorslaer (1997) is an exception). As for inference regarding various forms of stochastic dominance, it followed the same pattern as statistical inference on inequality measures. It focused on dominance tests in the context of income inequality literature namely in Anderson (1996), Davidson and Duclos (2000), Barrett and Donald (2003), Linton, Maasoumi and Whang (2005), Linton, Song and Whang (2010), Barrett, Donald and Batcharaya (2014) as well as Schechtman, Shelef, Yitzhaki and Zitifkis (2008). While Anderson (1996) test is based on the assumption that observations are drawn from two independent distributions, Davidson and Duclos's asymptotic approach to inference allows for observations to be drawn from a joint distribution. However, as in Anderson (1996), Davidson and Duclos's test uses a fixed number of arbitrary grid points.

The use of fixed number of arbitrary grid points is not a desirable feature of the test as the decision of the test will be contingent to the choice of the grid points and thus inconsistent (Barrett and Donald, 2003). To overcome this issue, Barrett and Donald (2003) propose a consistent Kolmogorov-Smirnov (KS) type test. Their approach tests dominance over all the points of the support, however, their test (as Anderson's) applies in cases where samples are drawn from independent distributions of income. Thus, Barrett and Donald's test does not allow for dominance for bivariate measures of inequality (i.e., for marginal conditional dominance). Schechtman et al. (2008) address this issue and propose a consistent statistical procedure in the context of a bivariate measure of inequality; the absolute concentration curve (a.k.a the generalized concentration curves in the health literature) in the context of portfolio choice in finance.

This paper contributes to this literature by proposing a consistent statistical test akin to the test Schechtman et al. (2008) and Barrett and Donald (2003) for the new dominance conditions introduced in this paper. Given that the dominance conditions we develop are for bivariate distributions, this paper is more akin to the work by Schechtman et al. (2008) than to the work of Barrett Donald (2003) and Linton et al. (2005). Although the hypothesis we are testing is in some cases mathematically analogous to the one tested in Schechtman et al. (2008), it remains different for three reasons. First, the framework and ethical principles are different. As a result, many of the welfare foundations and mathematical forms involved are different. Second, all estimators, dominance conditions and inference for indices obeying *pro-extreme rank* ethical principles are new. Finally, all the higher order dominance conditions for indices obeying higher order *pro-poor* ethical principles have no available statistical inference in the literature.

3 Measurement framework

The purpose of this section is to set the measurement framework and elaborate on the underlying ethical principles underlying health achievement and relative socioeconomic health inequality indices. These indices are functionals of the joint distribution of health, H and income, Y. In this paper, the term “income” refers to a measure of socioeconomic status.

Let H and Y be 2 random variables that are absolutely continuous with support on the positive half real line with densities f_H and f_Y respectively, with a joint density $f_{Y,H}$ and with a cumulative distribution of income, $F_Y(y)$.[1] We are interested in measuring health achievement and relative socioeconomic health inequality in a rank-dependent framework where ranks are individual's position in the distribution of socioeconomic statuses. In this context, a health achievement index is a weighted average that can be written as

$$A\left(h\right) = \int_0^1 \omega(p)h(p)\mathrm{d}p, \tag{1}$$

where $\omega(p)$ represents the social weight of an individual at rank $p \in [0,1]$ in the income distribution and $h(p)$ is the conditional expectation of health, H, with respect to Y equal to its p-quantile:

$$h(p) = E[H|Y = F_Y^{-1}(p)]. \tag{2}$$

In general, any relative index of socioeconomic health inequality can be interpreted as the ratio between the cost in health achievement associated with socioeconomic health inequalities and the average health status. This is why it is possible to write rank-dependent relative socioeconomic health inequality indices as a function of achievement indices. Thus, a relative index of socioeconomic health inequality can be directly be computed by taking the difference between the average health status, μ_h, and the health achievement index

[1] In this paper, we assume that this health measure is a ratio-scale variable. For ease of exposition we also assume that densities exist.

$A(h)$ divided by average health status:

$$I(h) = 1 - \frac{A(h)}{\mu_h} \tag{3}$$

where, $\mu_h = \int_0^1 h(p)\mathrm{d}p$ is the expectation of H. Formally, an index of relative socioeconomic health inequality can be rewritten as

$$I\left(h\right) = \int_0^1 \nu(p)\frac{h(p)}{\mu_h}\mathrm{d}p, \tag{4}$$

where the weight function $\nu(p) = 1 - \omega(p)$. The mathematical properties of the social weight function $\omega(p)$ and implicitly $\nu(p)$ are associated with indices' ethical principles. The following section will elaborate on two different ethical principles underlying socioeconomic health inequality and health achievement measures; the principle of income-related health transfer and the principle of symmetry around the median.

3.1 Principle of income-related health transfer

The following assumptions on the behaviour of $\omega^{(i)}(p) = \frac{\partial^i \omega(p)}{\partial p^i}$ $\forall\ i \in \{1, 2, \ldots, s-1\}$, and $\nu^i(p)$ will guarantee that the indices will have the properties that are compatible with the ethical principle deemed desirable for rank-dependent measures of health achievement and socioeconomic health inequality:

A.1 $\omega^{(1)}(p) \leq 0$

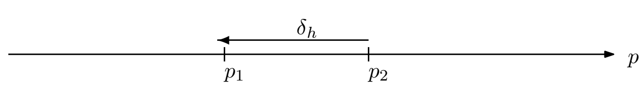
A.1' $\nu^{(1)}(p) > 0$ (i.e. $\omega^{(1)}(p) < 0$),

A.2 $\int_0^1 \omega(p)\mathrm{d}p = 1$,

$A(h)$ as defined in equation (1) is considered to be a rank-dependent measure of health achievement when the weight function, $\omega(p)$ satisfies assumptions A.1. and A.2. Similarly, $I(h)$ as defined in equation (4) is considered to be a rank-dependent measure of socioeconomic health inequality when the weight function $\nu(p)$ satisfies assumptions A.1' and

A.2. The role of assumption A.2 is to guarantee that the weight function $\nu(p)$ sums to zero (i.e., $\int_0^1 \nu(p)\mathrm{d}p = 0$) and thus that inequality indices have two fundamental desirable properties.[2] The first desired property requires that in the absence of health inequality (i.e., when everybody has the same health status, $\tilde{h}$), the inequality index $I(h)$ value be equal to zero. The second requires that $I(h)$ remains unchanged if everyone's health increases in the same proportion. The role of assumptions A.1 and A.1' is embedded in Bleichrodt and van Doorslaer (2006) *principle of income-related health transfer*. According to this principle, the contribution of an individual's health status to total health achievement (socioeconomic health inequality) is non-increasing (increasing) with socioeconomic status. This means that *ceteris paribus*, if the rich (poor) are relatively healthier, then the health achievement will be lower (higher), and the socioeconomic health inequality will be higher (lower). As illustrated in Figure 1, this principle implies that performing a mean preserving health transfer δ_h from an individual at socioeconomic rank p_2 to a person at a lower socioeconomic rank p_1, increases (decreases) health achievement (socioeconomic health inequality).

Figure 1: Second order ethical principle



It is important to highlight the interpretation of the slight difference between assumptions A.1 and A.1'. Assumption A.1 is less restrictive since it allows for $\omega(p) = 1$ for all p whereas assumption A.1' imposes a strict inequality. When $\omega^{(1)}(p) = 0$ for all p is combined with A.2, there is only one possible weight function, $\omega(p) = 1$, the resulting health achievement is the unweighted average health status μ_h.

Now that we have elaborated on the underlying ethical principle, we will use these

[2]Note that $\nu(p) = 1 - \omega(p)$.

assumptions to define the sets of all rank dependant health achievement and socioeconomic health inequality that we will be considering in this paper. Let us denote by Ω^2 the set of all rank-dependent health achievement indices and obeying assumptions A.1 and A.2. We can define this set as follows:

$$\Omega^2 := \left\{ A(h) \left| \begin{array}{c} \omega(p) \text{ is continuous and differentiable almost} \\ \text{everywhere over } [0,1], \int_0^1 \omega(p)\mathrm{d}p = 1, \\ \omega(1) = 0, \omega^{(1)}(p) \leq 0, \ \forall p \in [0,1] \end{array} \right. \right\}.$$

Let us denote by Λ^2 the set of all rank-dependent socioeconomic health inequality indices obeying assumptions A.1' and A.2. We can define this set as follows:

$$\Lambda^2 := \left\{ I(h) \left| \begin{array}{c} \nu(p) \text{ is continuous and differentiable almost} \\ \text{everywhere over } [0,1], \int_0^1 \nu(p)\mathrm{d}p = 0, \\ \nu(1) = 1, \nu^{(1)}(p) > 0, \ \forall p \in [0,1] \end{array} \right. \right\}.$$

3.2 Principle of symmetry around the median principle

According to Erreygers, Clarke and Van Ourti (2012), an index is considered a good measure of socioeconomic health inequality if it passes the *upside-down* test in addition to obeying the principle of income related health transfer. Let $g(p) = h(1-p)$, the *upside-down* test consists in verifying if $I(g(p))$ is always positive (negative) when $I(h(p))$ is negative (positive). Erreygers, Clarke and Van Ourti (2012) show that an index passes this test only if its weight functions $\nu(p)$ is symmetric around the median of socioeconomic statuses (i.e., around $p = 0.5$). This leads to the following additional assumptions on the behaviour of the social weight function:

A.3 $\nu(1-p) = -\nu(p)$,

A3' $\omega(1-p) = 2 - \omega(p)$.

Assumption A.3 also implies that $\nu(0.5) = 0$. Let $\Lambda^2_\rho \subset \Lambda^2$ be the subsets of rank-dependent socioeconomic health inequality indices that pass the *upside-down* test. It is possible to define these subsets as follows:

$$\Lambda^2_\rho := \left\{ I(h) \in \Lambda^2 \mid \nu(1-p) = -\nu(p) \ \forall p \in [0,1] \right\}.$$

As explained earlier, socioeconomic health inequality indices can always be expressed as a function of the achievement indices. This is why it is also possible to associate health achievement indices with these subsets of rank-dependent socioeconomic health inequality indices. Let $\Omega^2_\rho \subset \Omega^2$ be the subset of rank-dependent health achievement indices underlying socioeconomic health inequality indices that pass the *upside-down* test. Formally,

$$\Omega^2_\rho := \left\{ A_A(h) \in \Omega^2 \mid \omega(1-p) = 2 - \omega(p) \ \forall p \in [0,1] \right\}.$$

3.3 Examples of parametric class of indices

As pointed by Erreygers, Clarke and Van Ourti (2012), equation (4) is reminiscent of Mehran (1976) class of rank-dependent income inequality indices with a slight difference; individual ranks (socioeconomic status) are not determined by the rank of the variable of interest (health). The weight function $\omega(p)$ may take different functional forms that depend on socioeconomic rank p. Each subset of the class of achievement or inequality indices will depend on the specific form imposed on its weight function. For instance, if the analyst's ethical position is compatible with *sensitivity to poverty*, then a weight function $\omega(p) = \rho(1-p)^{\rho-1}$, where $\rho > 1$ the socioeconomic health inequality aversion parameter, is appropriate. In this case, equation (1) describes Wagstaff's (2002) class of health achievement indices, a subset of Ω^2. For the same specific parametric form of the weight function, equation (4) describes Wagstaff's (2002) class of extended health concentration indices a subset of Λ^2.[3] One may argue that an index is considered a good measure if it passes *upside-down test* (see, Erreygers, Clarke and van Ourti, 2012). In this case, the analyst values transfers occurring farther away from the median socioeconomic rank. As a result, *sensitivity to*

[3]Note that for the standard health concentration index $\rho = 2$.

extremities may be a more appropriate ethical position than *sensitivity to poverty*. A compatible weight function would be $\omega(p) = \beta 2^{\beta-2}\left[\left(p-\frac{1}{2}\right)^2\right]^{\frac{(\beta-2)}{2}}\left(p-\frac{1}{2}\right)$, where $\beta > 1$ is the socioeconomic health inequality aversion parameter. For this specific parametric weight function, equation (4) describes Erreygers, Clarke and Van Ourti's (2012) class of symmetric health socioeconomic inequality indices.[4] It is important to underline that the standard health concentration index (i.e., when $\rho = 2$) passes the *upside-down* test since $1-\rho(1-p)^{\rho-1} = 2p-1$ is by construction symmetric around the median. However, for all values of $\rho \neq 2$, the extended health concentration and health achievement indices do not pass the *upside-down* test. Wagstaff (2002) class of extended health concentration index and Erreygers, Clarke and Van Ourti's (2012) class of symmetric health socioeconomic inequality indices are both subsets of Λ^2. While Wagstaff (2002) and Erreygers, Clarke and Van Ourti (2012) are the most widely used indices in the health economics literature, they are not the only possible health achievement and socioeconomic health inequality indices. Other rank-dependent health achievement and socioeconomic health inequality indices based on similar ethical principles may be constructed.

4 Higher order aversion to socioeconomic health inequality

There are two distinct views in the literature on what constitutes a desirable higher order principle of aversion to socioeconomic health inequality. Wagstaff (2002) adopts a *pro-poor health transfer sensitivity* approach where health transfers occurring in the lower part of the distribution of socioeconomic ranks are deemed to be more desirable. Erreygers, Clarke and Van Ourti (2012) argue in favour of a *pro-extreme rank* transfer sensitivity approach where transfers occurring farther away from the median of socioeconomic ranks are valued more than transfers occurring closer to the median. In what follows we elaborate on the

[4]Note that when $\beta = 2$, the symmetric health socioeconomic inequality index collapses to the health concentration index.

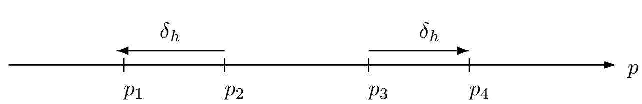
interpretation of each of these views and their implications on the behaviour of the social weight functions. We then define the sets of indices obeying these higher order ethical principles.

4.1 Generalized *pro-poor health transfer sensitivity* principles

The principle of transfer sensitivity was proposed by Kolm (1976) in the context of the income inequality literature. Mehran (1976) adapted this principle to a rank-dependent income inequality measurement framework. Mehran's rank-dependent version of the transfer sensitivity principle stipulates that a progressive transfer δ from an individual at socioeconomic rank p_2 to another one at rank p_1 (where $p_1 < p_2$) more than compensates for a regressive transfer of the same amount δ from an individual at rank p_3 to another at rank p_4 (where $p_4 > p_3 > p_2 > p_1$ and $p_4 - p_3 = p_2 - p_1$) provided there is no re-ranking following the transfers. One natural context in which one can extend this principle is the measurement of socioeconomic health inequality (as in Wagstaff, 2002).

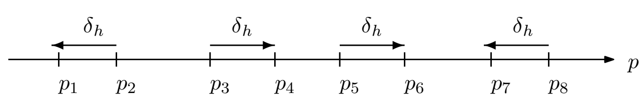
Wagstaff's (2002) class of indices (with $\rho > 2$) and all rank-dependent health achievement (socioeconomic health inequality) indices obeying assumptions A.1 (or A.1') and A.2, obey the *pro-poor health transfer sensitivity* principle if $\omega^{(2)}(p) \geq 0$ or $\nu^{(2)}(p) \leq 0$ for all $p \in [0, 1]$. This ethical principle stipulates that health achievement (socioeconomic health inequality) increases (decreases) when performing favourable composite mean preserving transfers. As illustrated in Figure 2, these transfers are such that a beneficial transfer of health δ_h from an individual at rank p_2 to an individual at rank p_1 and a reverse transfer of health δ_h from an individual at rank p_3 to an individual at rank p_4 increases (decreases) health achievement (socioeconomic health inequality).

Figure 2: Third order pro-poor transfer sensitivity principle



Higher-order pro-poor transfer sensitivity can be interpreted easily following Makdissi and Yazbeck (2014).[5] To understand how higher order transfer sensitivity operates, we need to revisit the second order principle (i.e., the transfer in Figure 1) and relate it to transfers occurring in Figure 2. The transfer in Figure 1 can be viewed as a favourable increase of δ_h for individual at rank p_1 combined with an unfavourable decrease of δ_h for the individual at rank p_2. The progressive transfer is then viewed as a combination of an improvement at rank p_1 and a deterioration at rank p_2. If we turn to Figure 2, the composite transfer can be decomposed as a favourable transfer between the individual at rank p_2 and the one at rank p_1 and an unfavourable transfer between the individual at rank p_3 and the one at rank p_4. Similarly to the third order *pro-poor transfer sensitivity* principal, Figure 3 illustrates a fourth-order with two pairs of transfers. A favourable combination of transfers occurring at lower socioeconomic ranks (between p_2 and p_1 and between p_3 and p_4) and an unfavourable one at higher socioeconomic statuses (between p_5 and p_6 and between p_8 and p_7). All rank-dependent health achievement (socioeconomic health inequality) indices obeying A.1 (or A.1') and A.2 obey this fourth order pro-poor transfer sensitivity principle if $\omega^{(2)} \geq 0$ (or $\nu^{(2)}(p) \leq 0$) and $\omega^{(3)} \leq 0$ (or $\nu^{(3)}(p) \geq 0$).

Figure 3: Fourth order pro-poor ethical principle



The generalized pro-poor transfer principle expands in a similar manner as the third and fourth order principles. In general, an index obeying A.1 (or A.1.') and A.2 obeys the sth-order pro-poor transfer sensitivity if $(-1)^i\omega^{(i)}(p) \geq 0$ or $(-1)^{i+1}\nu^{(i)}(p) \geq 0$ for all $i \in \{1, 2, \ldots, s-1\}$. More formally, the sets of indices of order s, Ω^s_π and Λ^s_π are the

[5] The authors adapt Fishburn and Willig (1984) generalized transfer sensitivity principles to the context of socioeconomic health inequality

sets of all rank-dependent indices obeying the principle of income-related health transfer and obeying all *pro-poor transfer sensitivity* principles of order $i \in \{3, 4, \ldots, s\}$. They are defined as follows:

$$\Omega_\pi^s := \left\{ A(h) \in \Omega^2 \;\middle|\; \begin{array}{c} \omega(p) \text{ is continuous and } (s-1)\text{-time differentiable almost} \\ \text{everywhere over } [0,1]\,, \omega^{(i)}(1) = 0, (-1)^i\, \omega^{(i)}(p) \geq 0,\ \forall p \in [0,1], \\ \forall i = 1, 2, \ldots, s-1 \end{array} \right\},$$

$$\Lambda_\pi^s := \left\{ I(h) \in \Lambda^2 \;\middle|\; \begin{array}{c} \nu(p) \text{ is continuous and } (s-1)\text{-time differentiable almost} \\ \text{everywhere over } [0,1]\,, \nu^{(i)}(1) = 0, (-1)^{i+1}\, \nu^{(i)}(p) \geq 0,\ \forall p \in [0,1], \\ \forall i = 1, 2, \ldots, s-1 \end{array} \right\}.$$

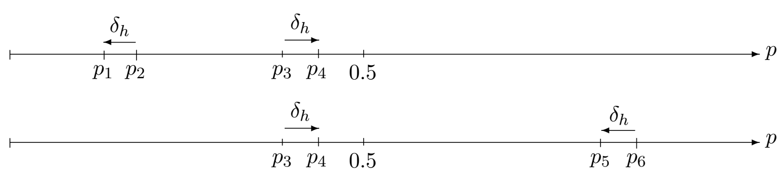
Note that increasing s means imposing more ethical structure on indices. This in turns implies that $\Omega_\pi^s \subset \Omega_\pi^{s-1} \subset \cdots \subset \Omega_\pi^3 \subset \Omega^2$, and $\Lambda_\pi^s \subset \Lambda_\pi^{s-1} \subset \cdots \subset \Lambda_\pi^3 \subset \Lambda^2$.

4.2 Generalized pro-extreme rank transfer sensitivity principles

Erreygers, Clarke and Van Ourti (2012) suggest that *pro-poor transfer sensitivity* is a debatable ethical principle when measuring socioeconomic health inequality. They argue that this principle is developed for a unidimensional (income) inequality context and cannot be readily applied to a bi-dimensional context (socioeconomic health inequality) where the aim is to measure the degree of association between health status and socioeconomic rank. More specifically, in the context of pro-poor transfer sensitivity ethical principle, a negative (positive) value of the index indicates that socioeconomic health inequalities are pro-poor (pro-rich) but this is regardless of the magnitude of the distance to the median socioeconomic rank. To account for this possibility, Erreygers, Clarke and Van Ourti (2012) introduce a symmetric class of indices without formalizing the associated ethical principles. This is why, we need to define a different transfer-sensitivity principles that are compatible with the *upside-down* test. Given that a class of indices that passes the *upside-down* test is more sensitive to transfers occurring farther away from the median of socioeconomic statuses (i.e., for cases when $\beta > 2$), we label the associated ethical principle as the *pro-extreme rank* transfer sensitivity principle. We also provide a formal presentation of the *pro-extreme rank* transfer sensitivity principles.

Figure 4 illustrates a 3rd order *pro-extreme rank* favourable combination of transfers. An index obeying 3rd order *pro-extreme rank* transfer sensitivity reacts favourably to a combination of favourable transfers occurring farther away from the median (p_2 to p_1 or p_6 to p_5) and an unfavourable one occurring closer to the median (p_3 to p_4). Order s *pro-extreme rank* transfer sensitivity principles impose an increasing weight to transfers occurring further away from the median of socioeconomic statuses as s increases. They are defined recursively by combining 2 types of transfer of order $s-1$, a favourable one occurring farther from the median and an unfavourable one occurring closer to the median of socioeconomic statuses.

Figure 4: Third order pro-extreme rank transfer sensitivity



In general, a rank-dependent socioeconomic health inequality (health achievement) index obeying A.1' (or A.1), A.2 and A.3 (or A.3') obeys the sth-order *pro-extreme rank* transfer sensitivity if $(-1)^{i+1}\nu^{(i)}(p) \geq 0$ ($(-1)^{i}\omega^{(i)} \geq 0$) for $p \in [0, 0.5]$ and for all $i \in \{1, 2, \ldots, s-1\}$. More formally, the sets of indices of order s, Λ_ρ^s and Ω_ρ^s are the set of all rank-dependent indices obeying the principle of income related health transfer and obeying all *pro-extreme rank* transfer sensitivity principles of order $i \in \{3, 4, \ldots, s\}$ are formally defined as follows:

$$\Lambda_\rho^s := \left\{ I(h) \in \Lambda_\rho^2 \left| \begin{array}{c} \nu(p) \text{is continuous and } (s-1)\text{-time differentiable almost} \\ \text{everywhere over } [0,1], \nu^{(i)}(0.5) = 0, (-1)^{i+1}\nu^{(i)}(p) \geq 0, \\ \forall p \in [0, 0.5],\ \forall i = 1, 2, \ldots, s-1 \end{array} \right. \right\},$$

$$\Omega_\rho^s := \left\{ A(h) \in \Omega_\rho^2 \left| \begin{array}{c} \omega(p) \text{is continuous and } (s-1)\text{-time differentiable almost} \\ \text{everywhere over } [0,1], \omega^{(i)}(0.5) = 0, (-1)^{i}\omega^{(i)}(p) \geq 0, \\ \forall p \in [0, 0.5],\ \forall i = 1, 2, \ldots, s-1 \end{array} \right. \right\}.$$

Once again, increasing s means imposing more ethical structure on indices. This implies that, $\Lambda^s_\rho \subset \Lambda^{s-1}_\rho \subset \cdots \subset \Lambda^3_\rho \subset \Lambda^2_\rho \subset \Lambda^2$ and $\Omega^s_\rho \subset \Omega^{s-1}_\rho \subset \cdots \subset \Omega^3_\rho \subset \Omega^2_\rho \subset \Omega^2$. In addition, because the higher order ethical principles are different, $\Lambda^s_\rho \cap \Lambda^s_\pi = \Omega^s_\rho \cap \Omega^s_\pi = \emptyset$ for $s \in \{3, 4, \dots\}$.

In this section, we have discussed the higher order ethical principles associated with the principle of income-related health transfer (i.e., aversion to socioeconomic health inequality). In doing so, we have presented two different higher order ethical principles: the generalized *pro-poor* health transfer sensitivity principle and the generalized *pro-extreme rank* transfer sensitivity principle. Since those two normative views co-exist in the literature, we will propose two tests that identify robust rankings, one for each of these views.

5 Identifying robust orderings of health distributions

In the previous section, we have discussed ethical principles that an analyst may impose on health achievement and socioeconomic health inequality indices. Whenever an analyst uses these indices to perform a comparison between two distributions, one important question surfaces: is the comparison obtained valid for wide spectra of indices obeying the same set of ethical principles? More specifically, is the comparison contingent to the particular mathematical expression of the index? To answer this question, one needs an approach that allows for comparisons that are robust over broad spectra of indices; this is why a dominance approach is necessary. In this section, we will first present some dominance tests developed in the Makdissi and Yazbeck (2014). These tests are based on the standard health concentration curves (Wagstaff, Paci and Van Doorslaer, 1991), generalized health concentration curves, s-health concentration curves and generalized s-health concentration curves. The health concentration curve may be used to identify orderings of distributions that are robust for all rank-dependent relative socioeconomic health inequality indices. Also, generalized health concentration curves may be used to identify robust orderings of health achievement indices. To identify robust orderings for subsets of relative socioeconomic health inequality and health achievement indices obeying *pro-poor* transfer sensitivity principles, the analyst can rely on s-health concentration curves and generalized s-health concentration curves respectively. Also, in this section, we propose new tests for the identification of robust orderings of distributions for indices obeying *pro-extreme rank* transfer sensitivity ethical principles. In doing so, we introduce new graphical tools: s-health range and generalized s-health range concentration curves. These curves are akin to the s-health concentration curves and generalized s-health concentration curves but are different as they are designed to obey the *symmetry around the median* principle and *pro-extreme rank* ethical principles.

5.1 Socioeconomic health inequality orderings

Before providing more details about these tests, it is important to provide some background on health concentration curves. Wagstaff, Paci and Van Doorslaer (1991) introduced the health concentration curve in the health economics literature. This curve plots the cumulative proportion of total health in the population against the cumulative proportion of individuals ranked by socioeconomic statuses. Formally, the health concentration curve, $C(p)$, is defined on the support $[0,1]$ as

$$C(p) = \frac{1}{\mu_h} \int_0^p h(u)\mathrm{d}u \tag{5}$$

When this curve lies above (under) the $45°$ diagonal, health inequality is pro-poor (pro-rich)[6]. An opposite conclusion may be reached if the analysis is based on an ill-health variable.

In addition to providing a graphical representation of the distribution of health statuses, Makdissi and Yazbeck (2014) explain how health concentration curves may be used to iden-

[6] In this context pro-poor means that the poor have better health than the rich

tify orderings of distributions that are robust for all rank-dependent relative socioeconomic health inequality indices.

Theorem 1 *Let $f^1_{Y,H}$ and $f^2_{Y,H}$ represent two joint densities of income and health. $I(h_1) \leq I(h_2)$ for all $I(h) \in \Lambda^2$ if and only if*

$$C_1(p) \geq C_2(p) \text{ for all } p \in [0, 1].$$

Theorem 1 is very powerful since it allows for the identification of orderings of the distribution that would remain the same for all rank-dependent relative socioeconomic health inequality indices. However, this robustness comes at a cost, as following such an approach produces an incomplete ordering of socioeconomic health distributions.[7]

When the ranking between two distributions is not robust, two paths may be followed. First, the analyst may decide to rely on a particular index by imposing a specific parametric form on the weight function (as seen in section 3.3). In this case, depending on whether the analyst's ethical position is compatible with *sensitivity to poverty* or *sensitivity to extremities*, the extended health concentration indices or the symmetric indices may be chosen. While this solution leads to complete orderings of distributions, this ordering is contingent to the ethical position adopted by the analyst and the specific mathematical structure of the chosen index.

An alternative solution is to increase the power of orderings by restricting the set of admissible rank-dependent relative socioeconomic health inequality indices either via *pro-poor* or via *pro-extreme rank* transfer sensitivity principles. It is important to note that these two sets of principles are based on different ethical views regarding what constitutes an increase in socioeconomic health inequality aversion (i.e., sensitivity to poverty and sensitivity to extremities). As a result, choosing one path or the other leads to different subsets of indices and may potentially lead to different orderings of distributions. In this

[7] For a complete proof see Makdissi and Yazbeck (2014).

case, the ordering will not depend on a specific parametric form of the weight function and therefore they will be robust. However, as in any dominance tests, the orderings will still be contingent to the ethical position taken by the analyst. In what follows, we will take the second path and develop tests that will identify robust orderings for both types of higher order ethical principles.

5.1.1 Pro-poor ethical principles

To test if orderings are robust for a subset of rank-dependent relative socioeconomic health inequality indices obeying these *pro-poor* ethical principles, Makdissi and Yazbeck (2014) have defined higher order s-health concentration curves, $C^s(p)$.[8] These are defined as:

$$C^s(p) = \int_0^p C^{s-1}(u)\mathrm{d}u, \tag{6}$$

where $C^2(p) = C(p)$. It is possible to identify robust rankings of distributions using these higher order health concentration curves.

Theorem 2 *Let $f^1_{Y,H}$ and $f^2_{Y,H}$ represent two joint densities of income and health. $I(h_1) \leq I(h_2)$ for all $I(h) \in \Lambda^s_\pi$ if and only if*

$$C^s_1(p) \geq C^s_2(p) \text{ for all } p \in [0,1].$$

Theorem 2 proposes a graphical test that is, once again, based on the non-intersection of two curves; the s-health concentration curves associated with the two distributions.[9] If there is an intersection between the two curves at order s, the analyst can impose more restriction on the subset of rank-dependent relative socioeconomic health inequality indices by imposing the pro-poor transfer sensitivity principle of order $s+1$. At the limit, when $s \to \infty$, a complete ranking is obtained. In this limit case, the test consists of comparing only $\lim_{p\to 0} \frac{h_1(p)}{\mu_{h1}}$ and $\lim_{p\to 0} \frac{h_2(p)}{\mu_{h2}}$.

[8]This curves adapt to the health inequality context the concept of s-concentration curves proposed by Makdissi and Mussard (2008) in the context of marginal indirect tax reforms.

[9]For a complete proof, please refer to Makdissi and Yazbeck (2014).

5.1.2 Symmetry around the median and pro-extreme rank ethical principles

As mentioned earlier, the analyst can choose to restrict the set of admissible rank-dependent relative socioeconomic health inequality indices by imposing *symmetry around the median* and *pro-extreme rank* transfer sensitivity. To test if orderings are robust for a subset of socioeconomic health inequality indices obeying these normative principles, we need to introduce a new graphical tool, the s-health range curves, $R^s(p)$. Let $r(p) = h(1-p) - h(p)$. These curves are formally defined as:

$$R^s(p) = \begin{cases} \frac{1}{\mu_h} \int_0^p r(u)\mathrm{d}u & \text{if } s = 2 \\ \int_0^p R^{s-1}(u)\mathrm{d}u & \text{if } s \in \{3, 4, \dots\} \end{cases} \tag{7}$$

As for the case of pro-poor ethical principle, robust rankings of socioeconomic health distributions can be identified using these health range curves.

Theorem 3 *Let $f^1_{Y,H}$ and $f^2_{Y,H}$ represent two joint densities of income and health. $I(h_1) \leq I(h_2)$ for all $I(h) \in \Lambda^s_\rho$ if and only if*

$$R^s_2(p) \geq R^s_1(p) \text{ for all } p \in [0, 0.5].$$

Theorem 3 provides a simple graphical test for the identification of robust orderings. Note that since $\Lambda^s_\rho \subset \Lambda^2$, the test based on $R^2(p)$ curves has more ordering power (is less general) than the test based on health concentration curves as in Theorem 1.[10] This increase in ordering power is obtained by imposing the principle of *symmetry around the median* on the indices. If the analysts think that a good relative socioeconomic health inequality index should pass the *upside-down* test, then he/she should use $R^2(p)$ curves, instead of health concentration curves $C(p)$. In this case, the only cost associated with the increase in the ordering power of the test is imposing symmetry of $\nu(p)$.

If there is an intersection between two health range curves at order s, the analyst can impose more restriction on the subset of rank-dependent socioeconomic health inequality

[10] For a complete proof, please refer to Appendix A1

indices by imposing the extreme rank transfer sensitivity principle of order $s+1$. At the limit, when $s \to \infty$, a complete ranking is obtained. In this limit case, the test consists of comparing only $\lim_{p\to 0} \frac{r_1(p)}{\mu_{h1}}$ and $\lim_{p\to 0} \frac{r_2(p)}{\mu_{h2}}$.

5.2 Health achievement orderings

Robust rankings of health achievement can be identified using the generalized health concentration curve. At quantile p, the generalized health concentration curve gives the absolute contribution of the p poorest individuals to average health. In other words, its value indicates the average health that would be attained if total health was only the sum of the health of these p poorest individuals. Formally, the generalized health concentration curve, $GC(p)$, is defined on the support $[0,1]$ as

$$GC(p) = \int_0^p h(u)\mathrm{d}u \tag{8}$$

Makdissi and Yazbeck (2014) explain how generalized health concentration curves may be used to identify orderings of distributions that are valid for all rank-dependent health achievement indices.

Theorem 4 *Let $f^1_{Y,H}$ and $f^2_{Y,H}$ represent two joint densities of income and health. $A(h_1) \geq A(h_2)$ for all $A(h) \in \Omega^2$ if and only if*

$$GC_1(p) \geq GC_2(p) \text{ for all } p \in [0,1].$$

Theorem 4 allows for the identification of health achievement orderings that remain valid for all rank-dependent health achievement indices.[11] As in the case of Theorem 1, this robustness comes at the cost of an incomplete order. As earlier, in case there is no dominance, two paths may be followed: choosing a particular index or imposing higher order ethical principles. As in the case of inequality indices, we will follow the second path.

[11] For a complete proof, please refer to Makdissi and Yazbeck (2014).

5.2.1 Pro-poor ethical principles

Let us first consider pro-poor transfer sensitivity principles. Makdissi and Yazbeck (2014) introduce s-generalized health concentration curves, $GC^s(p)$, for the identification of these robust orderings. These curves are defined on the support $[0,1]$ as

$$GC^s(p) = \int_0^p GC^{s-1}(u)\mathrm{d}u, \tag{9}$$

where $GC^2(p) = GC(p)$. Robust rankings of distributions can be identified using these higher order generalized health concentration curves.

Theorem 5 *Let $f^1_{Y,H}$ and $f^2_{Y,H}$ represent two joint densities of income and health. $A(h_1) \geq A(h_2)$ for all $A(h) \in \Omega^s_\pi$ if and only if*

$$GC^s_1(p) \geq GC^s_2(p) \text{ for all } p \in [0,1].$$

Theorem 5 proposes another graphical test based on the non-intersection of two curves, the s-generalized health concentration curves associated with the two distributions.[12] If there is an intersection between the two curves at order s, the analyst can impose more restriction on the subset of rank-dependent achievement indices by imposing the pro-poor transfer sensitivity principle of order $s+1$. At the limit, when $s \to \infty$, a complete ranking is obtained. In this limit case, the test consists of comparing only $\lim_{p\to 0} h_1(p)$ and $\lim_{p\to 0} h_2(p)$.

5.2.2 Symmetry around the median and pro-extreme rank ethical principles

An alternative path to imposing pro-poor ethical principle consists in imposing *symmetry around the median* and *pro-extreme rank* ethical principles to restrict the set of admissible health achievement indices. The identification of these orderings is based on a new graphical tool, the s-generalized health range curves, $GR^s(p)$. These curves are defined on the support

[12] For a complete proof, please refer to Makdissi and Yazbeck (2014)

[0, 0.5] as:

$$GR^s(p) = \begin{cases} \int_0^p r(u)\mathrm{d}u & \text{if } s = 2 \\ \int_0^p GR^{s-1}(u)\mathrm{d}u & \text{if } s \in \{3, 4, \dots\} \end{cases} \tag{10}$$

Robust rankings of socioeconomic health distributions can be identified using these health range curves.

Theorem 6 *Let $f^1_{Y,H}$ and $f^2_{Y,H}$ represent two joint densities of income and health. $A(h_1) \geq A(h_2)$ for all $A(h) \in \Omega^s_\rho$ if and only if*

$$GR^s_2(p) \geq GR^s_1(p) \text{ for all } p \in [0, 0.5].$$

and,

$$\mu_{h1} \geq \mu_{h2}$$

Theorem 6 offers another graphical test.[13] However, if we compare it with the test in Theorem 5, the identification of robust rankings for indices obeying the *symmetry around the median* and *pro-extreme rank* transfer principles has an additional condition on the average of health status when compared to pro-poor transfer principles.

6 Estimation and Inference

In this section, we show how to estimate the curves that provide the rankings that are robust to the class of indices chosen by the analyst. We then show how one can perform statistical inference on them.

6.1 Concentration and Range Curves estimators

Suppose we have a random sample of N individuals drawn from a joint distribution $f_{H,Y}$. We will first show how to construct estimators of C and R, C^s and R^s for $s > 2$ and then show how to test dominance using those curves. We start by showing that C and R can

[13]For a complete proof, please refer to Appendix A1.

both written in a form that is directly amenable to non-parametric estimation. First, note that $C(p)$ can be re-written as

$$C(p) = \frac{1}{\mu_h} \int_0^1 \mathbb{1}(u < p) h(u) \mathrm{d}u, \tag{11}$$

a simple estimator for $C(p)$ can be written as follows:

$$\widehat{C}(p) = \frac{1}{N\bar{h}} \sum_{i=1}^{N} h_i \mathbb{1}(y_i \leqslant \hat{F}_Y^{-1}(p)) \tag{12}$$

from a sample (y_i, h_i) for $i = 1, \ldots, n$. Here $\bar{h}$ is the sample average and $\hat{F}_Y^{-1}$ is a non-parametric estimator of the quantile function of Y based on the order statistics of (y_i). Estimators for $C^s(p)$ can be recursively derived from that of $C(p)$ (for details see Appendix A2.1). The resulting estimators are as follows:

$$\widehat{C}^s(p) = \frac{1}{N\bar{h}} \sum_{i=1}^{N} h_i \frac{(p - \hat{F}_Y(y_i))^{s-2}}{(s-1)!} \mathbb{1}(y_i \leqslant \hat{F}_Y^{-1}(p)) \tag{13}$$

The generalized concentration curve can be therefore written as:

$$\widehat{GC}^s(p) = \bar{h}\widehat{C}^s(p) \tag{14}$$

In a similar fashion, we can construct an estimator for R. Let us first rewrite $R(p)$ in the same form as $C(p)$:

$$\mu_h R(p) \quad = \quad \int_0^p r(u) \mathrm{d}u \tag{15}$$

Given that $r(u)$ can be written as $h(1-u) - h(u)$ for $u \in [0,1]$, we can re-write this relationship as follows:

$$\mu_h R(p) \quad = \quad \int_{1-p}^1 h(u)\mathrm{d}u - \int_0^p h(u)\mathrm{d}u, \tag{16}$$

which can be re-written as follows:

$$\mu_h R(p) = \int_0^1 [\mathbb{1}(u > 1-p) - \mathbb{1}(u < p)] h(u) \mathrm{d}u \tag{17}$$

A simple estiamtor for $R(p)$ can be written as follows:

$$\widehat{R}(p) = \frac{1}{N\bar{h}} \left\{ \sum_{i=1}^{N} h_i[\mathbb{1}(y_i > \hat{F}_Y^{-1}(1-p))] - \sum_{i=1}^{N} h_i[\mathbb{1}(y_i \leqslant \hat{F}_Y^{-1}(p))] \right\} \tag{18}$$

Similarly to C^s, it is possible to recursively compute estimators of R^s by first plugging the estimators of $\widehat{R}^{s-1}$, integrating them analytically and then by recursively computing $\widehat{R}^s$ (see details in Appendix A2.2). The resulting estimators are as follows:

$$\begin{aligned} \widehat{R}^s(p) &= \frac{1}{N\bar{h}} \sum_{i=1}^{N} h_i \frac{1}{(s-1)!} p^{s-2}(p + (s-1)[\hat{F}_Y(y_i) - 1])[\mathbb{1}(y_i > \hat{F}_Y^{-1}(1-p))] \\ &- \frac{1}{N\bar{h}} \sum_{i=1}^{N} h_i \frac{(p - \hat{F}_Y(y_i))^{s-2}}{(s-1)!}[\mathbb{1}(y_i \leqslant \hat{F}_Y^{-1}(p))] \end{aligned} \tag{19}$$

The generalized range curve can be written as follows:

$$\widehat{GR}^s(p) = \bar{h}\widehat{R}^s(p) \tag{20}$$

6.2 Dominance tests

Let us denote by L one of the curves from the previous section (e.g., $C(p)$). Let L_1 and L_2 two different theoretical curves (corresponding to two different theoretical populations). Assume that we have an i.i.d. sample of size n_1 from the random variable corresponding to first theoretical curve L_1 and an i.i.d. sample of size n_2 from the random variable corresponding to the second theoretical curve L_2. Denote those samples by $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. As we are interested in testing the dominance between two distributions, we define the new function $L_{12}(p) := L_1(p) - L_2(p)$ for $p \in [0, 1]$. The null and alternative hypotheses we interested in are:

$$\begin{aligned} H_0 &: L_{12}(p) \leqslant 0, \forall p \\ H_1 &: L_{12}(p) > 0 \text{ for some } p \end{aligned}$$

When performing inference, for each pair of distributions we will test a set of inequalities. In this paper, we will test for H_0: $L_{12} \leq 0$ for all p and H_0: $L_{12} \geq 0$ for all p where under the null we assume dominance. Because no statistic can distinguish between weak and strict dominance, the test focuses on weak dominance. If we can reject one of the null hypotheses of dominance for the same pair of distributions, then we have evidence against that null of the dominance of one distribution over the other. While one may think that it is more intuitive to test the null hypothesis of non-dominance and hence establish a case of dominance, such a test requires a strong evidence against the null, which may be difficult to obtain over the entire support (Davidson and Duclos, 2013).

The nonparametric estimators $\hat{L}_1$ and $\hat{L}_2$ of L_1 and L_2 respectively can be constructed from those two samples and $\hat{L}_{12}(p) = \hat{L}_1(p) - \hat{L}_2(p)$. Let $\tau = \sup_p L_{12}(p)$, it is straightforward to construct a KS type test statistic $\hat{\tau}$ that is a non-parametric estimator of τ as follows:

$$\hat{\tau} = \sqrt{\frac{n_1 n_2}{n_1 + n_2}} \sup_p \hat{L}_{12}(p). \tag{21}$$

The asymptotic distribution of $\hat{\tau}$ will be that of a functional of two-dimensional Gaussian process that is very complicated to compute. To overcome this issue, we will rely on a bootstrap procedure as in Shechtman et al. (2008). For a detailed description of the bootstrap procedure, please refer to the Appendix A3.

As for the indices obeying the *symmetry around the median principle* and *pro-extreme rank* principles (i.e., theorem 6), the associated the joint test H_0^1 and H_0^2 can be defined as follows,

$$\begin{aligned} H_0^1 &: GR_{12}(p) \leqslant 0, \forall p \\ H_1^1 &: GR_{12}(p) > 0 \text{ for some } p, \end{aligned}$$

and,

$$H_0^2 \quad : \quad \mu_1 \geq \mu_2$$
$$H_1^2 \quad : \quad \mu_1 < \mu_2,$$

Where the nonparametric estimators $\widehat{GR}_1$ and $\widehat{GR}_2$ of GR_1 and GR_2 respectively can be constructed from those two samples and $\widehat{GR}_{12}(p) = \widehat{GR}_1(p) - \widehat{GR}_2(p)$. The test statistic, $\hat{\tau}$, to test the H_0^1 for both tests takes the same form, however, the joint test (i.e., H_0^1 and H_0^2) has an additional condition on the mean that needs to be tested when establishing the dominance results. To account for this additional test, we adjust the significance level of the joint test by relying on the Holm procedure as described in the Lehmann and Romano (2005) [chapter 9 p. 348]. The purpose of the procedure is to control for the family-wise error rate (FWER), which is the probability of one or more false rejections not exceeding a certain level, by making sure that this error is below a certain threshold α.

Let $I \subset \{1, 2\}$ be the set of indices for which H_0^i is true for $i = 1, 2$, then the objective is to make $\Pr\left\{\textit{reject any } H_0^i \textit{ with } i \in I\right\} \leqslant \alpha$. Given the two tests H_0^1 and H_0^2 with p-values p_1 and p_2, the Holm procedure works as follows. First, order the p-values $p_1 \leqslant p_2$ and label the correspondingly H_0^1 and H_0^2. If $p_1 \geqslant \frac{\alpha}{2}$, then we do not reject both hypotheses and stop. However, if $p_1 < \frac{\alpha}{2}$ and $p_2 \geqslant \alpha$, then we reject H_1 and do not reject H_0^2. Otherwise, reject both hypotheses. It should be noted that if we reject one of the two hypothesis, then we reject dominance.

7 Empirical illustration

To provide evidence that the differences between different ethical principles adopted by the analyst influence the type conclusion reached, we conduct an empirical illustration of the methods proposed using National Health Interview Survey data from years 1997 and 2014. We will focus on comparisons of two ill-health variables that have been of great interest in the health economics literature: cigarettes consumption (i.e., the number of cigarettes/day) and overweightedness (defined as max[0,BMI-25]). Given that the empirical application is mainly for illustration purposes, we will refrain from drawing policy recommendation, but we will indicate potential interesting paths.

The NHIS monitors health outcomes of Americans since 1957. It is a cross-sectional household interview survey representative of American households and non-institutionalized individuals. It contains data on a broad range of health topics that are collected via personal household interviews. For comparison purposes, we will focus 1997 and 2014 public use data and restrict our attention to the adult population. After applying all these restrictions to the data, we end up with a sample size is 34776 for overweightedness and 35667 for cigarette consumption in 1997 and is 35197 for overweightedness and 36363 for cigarette consumption in 2014. We will use the sample adult file to extract information on health-related behaviour and use family income adjusted for family size to infer the socioeconomic rank of individuals.[14]

We will first start the illustration by looking at comparisons from an inequality perspective then we will revisit these comparisons from an achievement perspective.

7.1 Comparisons of inequalities in health related behaviours and outcomes

In the first set of inequalities comparisons presented in Table 1 we will focus on comparisons at the national level. These comparisons will be complemented by regional comparisons in Table 2. Cigarette consumption seems to display a higher socioeconomic health inequality in 2014 than in 1997 (Figure 5). There is a clear dominance of the concentration curve C^2_{2014} over C^2_{1997} without any intersection on the support. As is shown in Table 1 when the null hypothesis of the dominance of C^2_{1997} over C^2_{2014} there is a very weak evidence against

[14]We compute equivalent income by dividing family income by the square root of household size.

the null. However, when the null hypothesis of the dominance of C^2_{2014} over C^2_{1997} is tested, there is strong evidence against the null (p-value=0.0000). As a result, one can conclude that there is more socioeconomic health inequality in smoking in 2014 for all indices obeying the income-related health transfer principle. While deriving any policy conclusion is beyond the scope of this paper, it is important to underline that an increase in the disparities at the cigarette consumption level may be a major contributor to the widening disparities in health outcomes.

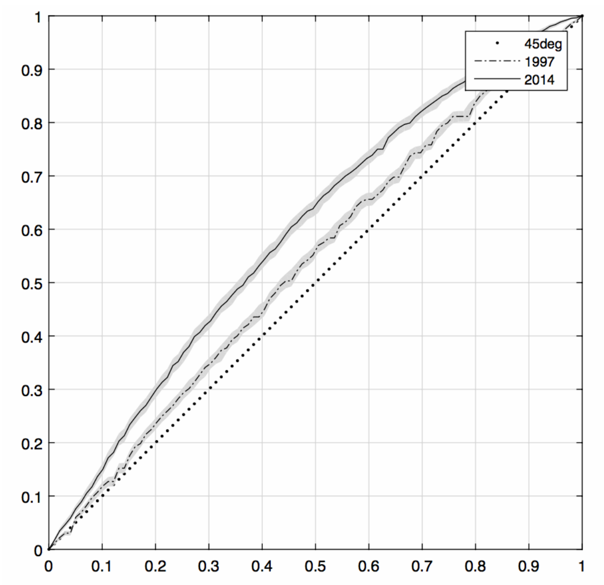


Figure 5: C^2 comparison: cigarette consumption

In addition to testing dominance at the second order (i.e., C^2) we provide a test for order 3 dominance, C^3, and order 2 dominance for indices meeting the *upside-down* test criteria (i.e., ranges curves R^2). We know from our theoretical results that if dominance is obtained at the second order for the C^2, it follows that dominance will be obtained for both higher order dominance C^s and second order range curves R^2. To show the empirical validity of these theoretical results we conduct this additional test. Test results presented in the lower panel of Table 1 confirm what was theoretically expected; there is more socioeconomic

Table 1: Dominance tests for C^s and R^s comparisons for cigarette consumption and overweightedness

	p-value			
	cigarette cons.		overweightedness	
	s=2	s=3	s=2	s=3
$H_0 : C^s_{1997}(p) \leqslant C^s_{2014}(p), \forall p$ $H_1 : C^s_{1997}(p) > C^s_{2014}(p)$ for some p	0.9970	0.8248	0.0060	0.0010
$H_0 : C^s_{2014}(p) \leqslant C^s_{1997}(p), \forall p$ $H_1 : C^s_{2014}(p) > C^s_{1997}(p)$ for some p	0.0000	0.0000	0.0000	0.8658
$H_0 : R^s_{1997}(p) \leqslant R^s_{2014}(p), \forall p$ $H_1 : R^s_{1997}(p) > R^s_{2014}(p)$ for some p	0.0000		0.5305	
$H_0 : R^s_{2014}(p) \leqslant R^s_{1997}(p), \forall p$ $H_1 : R^s_{2014}(p) > R^s_{1997}(p)$ for some p	0.9670		0.0020	

health inequality in smoking in 2014 than 1997. This is true for all indices obeying the income-related health transfer principle and for the subset of these indices obeying the pro-poor transfer sensitivity principle. Similarly, as there is more socioeconomic health inequality in smoking in 2014 for all indices obeying the income-related health transfer principle, this result applies to the subset of these indices passing the *upside-down* test.

Another health variable that one may want to consider in the analysis of socioeconomic health inequality is overweightedness (defined as $\max[0, BMI - 25]$). Looking at the top left panel in Figure 6, one can notice that the two overweightedness concentration curves (C^2) intersect. We, therefore, cannot reject the null hypothesis that they are equal as there is strong evidence against the null when dominance of C^2_{1997} over C^2_{2014} and dominance of C^2_{2014} over C^2_{1997} are tested at the 1% level (p-values are respectively 0.0060 and 0.0000 in Table-1).[15] As a result, we cannot assess whether socioeconomic health inequality in overweightedness has increased or decreased when we consider all indices obeying the

[15]It is important to note that if one decreases the level to 0.5%, the dominance conclusions reached at order 2 will not hold at third order. While this may seem in contradiction with the theory at first, it is not the case in this application. In reality this “incoherence” between the second and third order dominance is due to the magnitude of the distance just before $p = 0.8$ in the first panel of Figure 6. Given that this distance is quite large, integrating over the support of the second order curves results apparently flipped around result at the third order.

income-related health transfer principle. As mentioned earlier in the paper, when there is no clear dominance in the context of concentration curves (i.e., C^2), one can consider the subset of indices obeying a higher order ethical such as the pro-poor transfer sensitivity principle (i.e. considering, C^3) so we follow this path. The results shown in the top right panel of Figure 6 allow for the conclusion that socioeconomic health inequality in overweightedness decreases from 1997 to 2014. In other words, there is more socioeconomic health inequality in overweightedness in 1997 for all indices obeying the income-related health transfer principle as well as the pro-poor transfer sensitivity principle. An alternative path may be taken in the absence of dominance at the second order if one is willing to restrain the set of indices considered to the subset of indices passing the *upside-down* test (i.e., R^2 curves). As shown in the lower panel of Figure 6, following this paths leads to the same conclusion as the one reached when exploring higher order dominance in the case of concentration curves. Indeed, there is more socioeconomic health inequality in overweightedness in 1997 for all indices obeying the income-related health transfer principle and the *upside-down* test and these results are supported by the associated p-values displayed in Table 1. It is important to note that the subset of indices obeying higher order *pro-poor* principles and the subset of indices that pass the *upside-down* test are disjoint. So while the conclusions reached in this empirical application are the same, there is no reason for this to be always the case.

As for regional comparisons, we focus our attention on the most recent year, 2014 and compare the Northeast, the West, the Midwest and the South.[16] When we focus on cigarette consumption, we notice there are no clear patterns of dominance and thus no complete order.[17] More specifically, at 5% significance level, the West dominates the Northeast for all indices obeying *pro-poor* transfer sensitivity (i.e., Λ^3_π), and dominates the

[16]It is important to note that we chose the most recent year to save on space.

[17]It is important to note that the rows have lower socioeconomic inequality than the columns.

Figure 6: Socioeconomic inequality in Obesity/Overweight

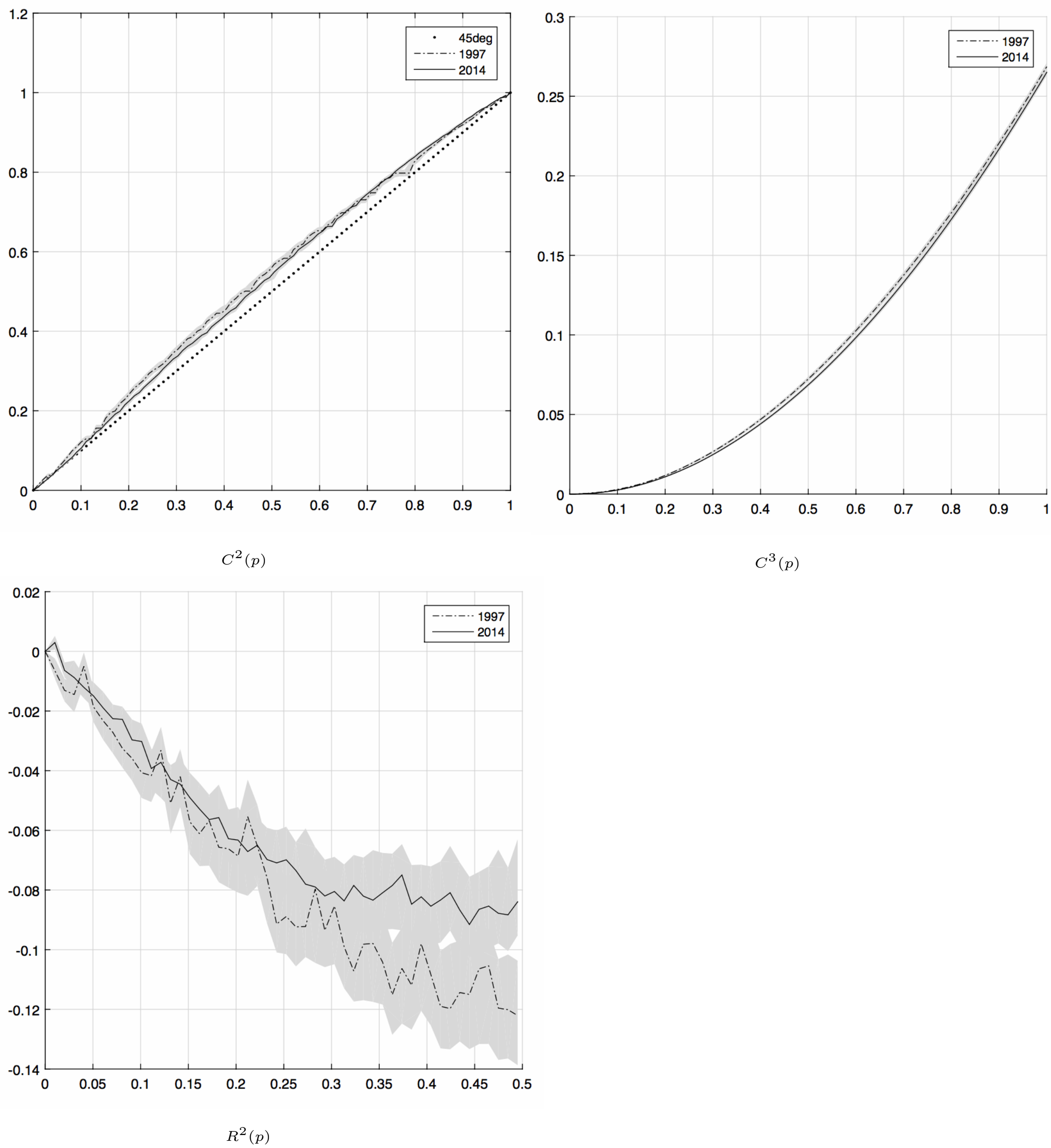


Table 2: Regional dominance tests: cigarette consumption and overweightedness

Cigarette consumption				
	Northeast	West	Midwest	South
Northeast	-		ND	ND
West	Λ_{π}^{3} ** and Λ_{ρ}^{2} **	-	Λ_{ρ}^{2} **	Λ^{2} ***
Midwest	ND		-	ND
South	ND		ND	-
overweightedness				
	Northeast	West	Midwest	South
Northeast	-			
West	Λ^{2} ***	-	ND	ND
Midwest	Λ^{2} ** and Λ_{π}^{3} *** and Λ_{ρ}^{2} ***	ND	-	ND
South	Λ^{2} ** and Λ_{ρ}^{2} ***	ND	ND	-

Significance levels ** 5%; *** 1%

South for all indices (i.e., Λ^2). Also, for the subset passing the *upside-down* test, the West dominates the Northeast, the Midwest at the second order. If increase the significance level to 1%, then we only have one dominance result: the West dominates the South at the second order for all indices (i.e., Λ^2). Turning our attention to regional dominance in the case of overweightedness, we notice that the Northeast is dominated by the West, the Midwest as well as the South. While the significance level and the order at which this dominance occur vary by region, one can safely say that this dominance occurs at the 5% significance level. If we were to increase the significance level to 1%, then the order and subset of indices at which this dominance occurs changes. For instance, the Northeast is dominated by the Midwest at the *pro-poor* third order instead of the second order. Also, the Northeast is dominated by the South and the Midwest for all subset of indices passing the *upside-down* test.

7.2 Comparisons of health achievements in health related behaviours and outcomes

In this section, we will be comparing health achievements between 1997 and 2014. To save on space, Table 3 we will not report the p-values but rather report dominance along with the standard notation to indicate the significance level of the dominance tests. To read Table 3, one has to keep in mind that the columns dominate the rows; this means that when there is dominance, the year that dominates has lower "ill health" level and hence higher health achievement. Given that we are dealing with an "ill-health" variable it is more sensible to talk about health failures rather than health achievements (see Makdissi, Sylla and Yazbeck, 2013).[18]

Comparisons of generalized health concentration curve reveal that GC^2_{2014} dominates GC^2_{1997} with strong evidence against the null hypothesis. This means that, as far as cigarette consumption is concerned, there is more health failure in 1997 than in 2014 for all health achievement(/failure) indices obeying the principle of income-related health transfer. As in the case of inequality, we reach the same conclusion if we test for a higher order dominance (i.e., GC^3). This, once again, confirms what was theoretically expected. In other words, since there is more health failure (when is considered smoking) in 1997 for all indices obeying the income-related health transfer principle, it is expected this is true for the subset of these indices that are obeying the pro-poor transfer sensitivity principle. As mentioned earlier, the analyst may argue that the principle of pro-poor transfers sensitivity is debatable and focus on the set of indices that pass the *upside-down* test. To account for this possibility, we test for dominance using the generalized health range curves. Empirical results show that there is more failure in smoking in 1997 for all indices obeying the income-related health transfer principle and the *upside-down* test. Given that the set of indices that pass

[18] It is important to note that the rows have lower failure than the columns in the tables which means that the rows have a higher achievement.

Table 3: Evolution of health failures: 1997-2014

	1997	2014
1997	-	overweightedness: Ω^2 ***
2014	Cigarette: Ω^2 ***	-

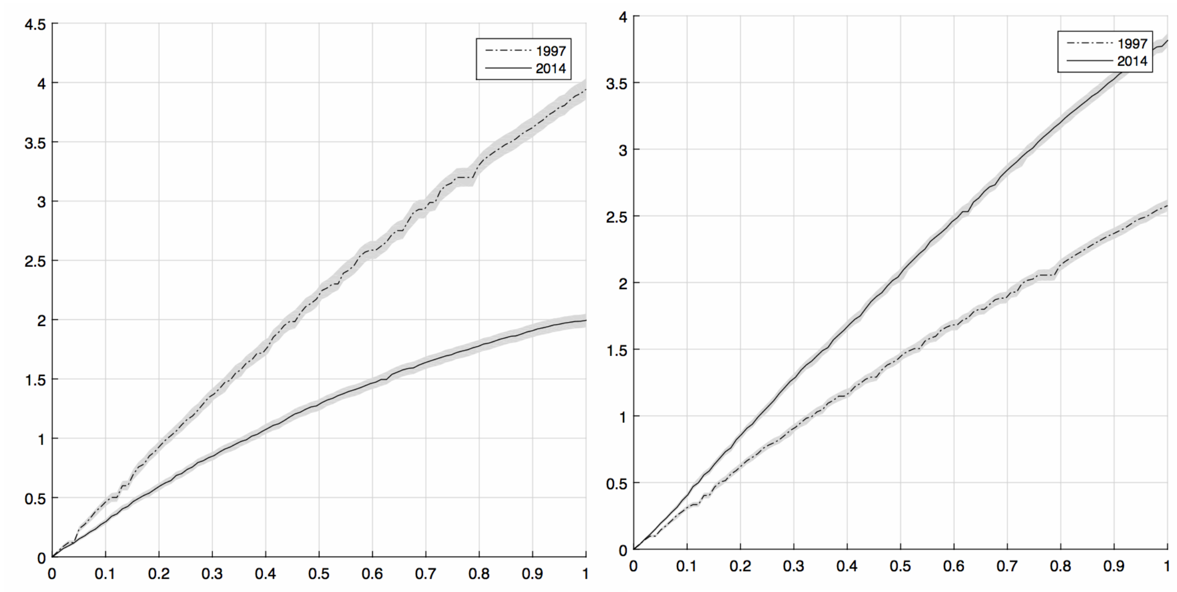
Significance levels: ** 5%; *** 1%.

the *upside-down* test are subsets of the indices belonging to Ω^2, we can re-write the results concisely by saying that there is a dominance at the second order for all rank-dependent indices that is all indices in Ω^2. As for overweightedness, we have the mirror picture of the cigarette consumption comparison as there is more failure in 2014 than in 1997. These second order dominance results are statistically significant at the 1% level and hold for all rank-dependent indices (see Table 3).

Before turning to the regional comparisons, it is important to compare the results obtained from the inequality analysis with the results obtained from the achievement analysis to emphasize the policy relevance of developing and using both approaches in an inequality analysis. While the inequality analysis revealed that there is more inequality in cigarette consumption in 2014, the analysis on health achievement (or failure) shows that there is a lower failure in health in 2014 than 1997. So while the inequality analysis may show that there are concerns regarding socioeconomic inequality in smoking behaviour, this same behaviour seems to be less prevalent when assessed by a measure that puts higher weight for smoking behaviour when it occurs in the lower part of the income distribution. The same logic applies to the results obtained in overweightedness. The socioeconomic inequalities are lower in 2014 than in 1997 but the failure is higher in 2014 than in 1997. The results discussed in this section indicate that focusing on inequality alone provides an incomplete picture of the situation.

As for regional comparisons for health failures, it is clear that we have more results than in the inequality section. The first panel of Table 4 focuses on cigarette consumption

Figure 7: Second order health failures comparisons



Cigarette

overweightedness

Table 4: Regional dominance tests: cigarette consumption and overweightedness

	Northeast	West	Midwest	South
Cigarette consumption				
Northeast	-		Ω^2 ***	Ω^2 ***
West	Ω^2 ***	-	Ω^2 ***	Ω^2 ***
Midwest			-	
South			Ω^2 ***	-
overweightedness				
	Northeast	West	Midwest	South
Northeast	-		Ω^2 ***	Ω^2 ***
West	Ω^2 ** and Ω^2_ρ ***	-	Ω^2 ***	Ω^2 ***
Midwest			-	ND
South			ND	-

Significance levels ** 5%; *** 1%

in 2014 and shows evidence that we have dominance results for all region at the 1% level. This allows for a complete ordering of regions in ascending inequalities as follows West, Northeast, South and Midwest. The second panel in Table 4 shows second order dominance results for overweightedness at mixed significance levels (i.e., some are at the 5% level and others are at the 1% level). Unlike the case of cigarette consumption, we do not have a complete ordering of regions for overweightedness. All we can say is that the West dominates the Northeast, the Midwest and the South and that the Northeast dominates the Midwest and South. To assess whether we can have a dominance result at a higher significance level (i.e., 1% level instead of 5% level) for the Northeast and West, we follow the path followed in the inequality section by focusing on indices that pass the *upside-down* text. In doing so, one needs to remember that testing for achievement (/failure) for these subsets of indices requires a joint test on the range curves and the average value of the health variable. Figure 8 displays GR^2 curves for the two regions where GR^2_W seem to be everywhere above (or equal) to GR^2_{NE}. The results of the associated statistical tests displayed in Table 5 suggest that we cannot reject dominance and that the West has less failure in overweightedness than the Northeast if we focus our attention on indices that pass the *upside-down* test and obey the *principle of income-related health transfer*.

8 Conclusion

In this paper, we adopted a unified approach to indices obeying *pro-poor ethical* principles as well as the *symmetry around the median* ethical principle and *pro-extreme rank* principles. To do so, we first fill the gap in the literature by formalizing the ethical principles associated with the symmetric indices (i.e., the set of indices that pass the *upside-down* test). We coin these ethical principles as the *symmetry around the median* ethical principle and *pro-extreme rank* ethical principles. We then develop the curves associated with these principles, the

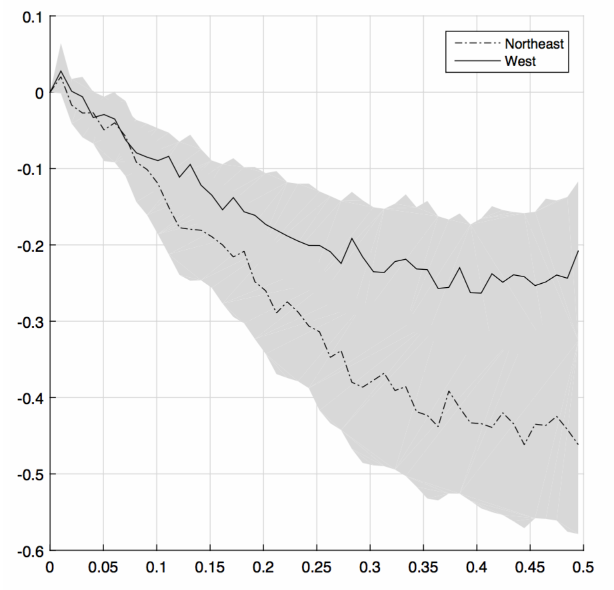


Figure 8: GR^2 comparison: overweightedness

Table 5: Dominance tests for failure in overweightedness between the Northeast and the West for indices belonging to Ω_ρ^2

	p-value
$H_0 : GR^2_W(p) \leqslant GR^2_{NE}(p), \forall p$ $H_1 : GR^2_W(p) > GR^2_{NE}(p)$ for some p	0.0040
$H_0 : GR^2_{NE}(p) \leqslant GR^2_W(p), \forall p$ $H_1 : GR^2_{NE}(p) > GR^2_W(p)$ for some p	0.9219
$H_0 : \mu_W \geqslant \mu_{NE}$ $H_1 : \mu_W < \mu_{NE}$	0.2843
$H_0 : \mu_{NE} \geqslant \mu_W$ $H_1 : \mu_{NE} < \mu_W$	0.7157
$H_0 : \mu_W = \mu_{NE}$ $H_1 : \mu_W \neq \mu_{NE}$	0.5876

health range curve and the s-health range curves, and derive the dominance conditions that allow us to identify robust orderings of joint distributions of income and health. Having filled the gap in the inequality measurement literature, we proceed to the literature on the statistical inference and provide the natural estimators for the indices obeying both *pro-poor* and *pro-extreme rank* ethical principles. Based on these estimators and on the work of Linton et al. (2005) and Schechtman et al. (2008) we develop KS-type statistical tests associated with the dominance tests for indices obeying both ethical principles. Finally, to illustrate the applicability of the methods proposed we provide an empirical illustration using information on overweightedness and cigarette consumption from the NHIS 1997 and 2014.

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Appendix

A1 Proofs for section 4

Proofs of Theorems 1, 2, 4, 5 are provided by Makdissi and Yazbeck (2014).

Proof of Theorem 3. First note that for $I(h) \in \Lambda_\rho^s$, equation (4) can be rewritten as

$$I(h) = -\frac{1}{\mu_h} \int_0^{0.5} \nu(p) r(p) \mathrm{d}p \tag{A1}$$

Integrating by parts equation (A1), we get

$$I(h) = -\nu(p) R^2(p) \Big|_0^{0.5} + \int_0^{0.5} \nu^{(1)}(p) R^2(p) \mathrm{d}p. \tag{A2}$$

Since by definition $R^2(0) = 0$ and $\nu(0.5) = 0$ for all indices $I(h) \in \Lambda_\rho^s$, the first term on the right hand side of the equation is nil. This yields to

$$I(h) = \int_0^{0.5} \nu^{(1)}(p) R^2(p) \mathrm{d}p. \tag{A3}$$

Now assume that for $s-1$, we have

$$I(h) = (-1)^{s-1} \int_0^{0.5} \nu^{(s-2)}(p) R^{s-1}(p) \mathrm{d}p. \tag{A4}$$

Integrating by parts equation (A4) yields

$$I(h) = (-1)^{s-1} \left\{ \nu^{(s-2)}(p) R^s(p) \Big|_0^{0.5} - \int_0^{0.5} \nu^{(s-1)} R^s(p) \mathrm{d}p \right\}. \tag{A5}$$

Since by definition $R^s(0) = 0$ and $\nu^{(s-2)}(0.5) = 0$ for all indices $I(h) \in \Lambda_\rho^s$, the first term in the braces on the right hand side of the equation is nil. This yield

$$I(h) = (-1)^s \int_0^{0.5} \nu^{(s-1)}(p) R^s(p) \mathrm{d}p. \tag{A6}$$

Given that equations (A3) and (A6) both conform to the relation depicted in equation (A4), it follows that equation (A6) holds for all $s \in \{2, 3, \dots\}$. Let $\Delta I_{12} = I(h_2) - I(h_1)$. From equation (A6), we get

$$\Delta I_{12} = (-1)^s \int_0^{0.5} \nu^{(s-1)}(p) \left[R_2^s(p) - R_1^s(p) \right] \mathrm{d}p. \tag{A7}$$

Note that $(-1)^s \nu^{(s-1)}(p)$ is non negative. This implies that if $R_2^s(p) \geq R_1^s(p)$ for all $p \in [0, 0.5]$, then $\Delta I_{12} \geq 0$. This proves for sufficiency of the condition.

Having provided a sufficiency condition let us now prove for the necessity of the condition. Consider now the set of indices $I(h) \in \Lambda_\rho^s$ for which $\nu^{(s-2)}(p)$ takes the following form:

$$\nu^{(s-2)}(p) = \begin{cases} (-1)^{s-1}\varepsilon & 0 \leq p_c \\ (-1)^{s-1}\left[p_c + \varepsilon - p\right] & p_c \leq p \leq p_c + \varepsilon \\ 0 & p \geq p_c + \varepsilon \end{cases} \tag{A8}$$

where $p_c \in [0, 0.5]$. Since $\nu(p)$ is differentiable almost everywhere, it satisfies the conditions in the definition of Λ_ρ^s. Differentiating equation (A8) yields

$$\nu^{(s-1)}(p) = \begin{cases} 0 & 0 \leq p_c \\ (-1)^s & p_c \leq p \leq p_c + \varepsilon \\ 0 & p \geq p_c + \varepsilon \end{cases} \tag{A9}$$

Imagine now that $R_2^s(p) < R_1^s(p)$ on an interval $[p_c, p_c + \varepsilon]$ for ε that can be arbitrarily close to 0. For any $\nu(p)$ obeying the relation in equation (A8), the expression in equation (A7) is negative. Hence it cannot be that $R_2^s(p) < R_1^s(p)$ for $p \in [p_c, p_c + \varepsilon]$. This proves the necessity of the condition. ■

Proof of Theorem 6. First note that for $A(h) \in \Omega_\rho^s$, equation (1) can be rewritten as

$$A(h) = \int_0^1 (1 - \nu(p))h(p)\mathrm{d}p \tag{A10}$$

$$A(h) = \mu_h - \int_0^1 \nu(p)h(p)\mathrm{d}p \tag{A11}$$

$$A(h) = \mu_h + \int_0^{0.5} \nu(p)r(p)\mathrm{d}p \tag{A12}$$

Integrating by parts equation (A12), we get

$$A(h) = \mu_h + \nu(p)GR^2(p)\big|_0^{0.5} - \int_0^{0.5} \nu^{(1)}(p)GR^2(p)\mathrm{d}p. \tag{A13}$$

Since by definition $GR^2(0) = 0$ and $\nu(0.5) = 0$ for all indices $A(h) \in \Omega_\rho^s$, the second term on the right hand side of the equation is nil. This yields to

$$A(h) = \mu_h - \int_0^{0.5} \nu^{(1)}(p)GR^2(p)\mathrm{d}p. \tag{A14}$$

Now assume that for $s-1$, we have

$$A(h) = \mu_h + (-1)^{s-2} \int_0^{0.5} \nu^{(s-2)}(p) GR^{s-1}(p) \mathrm{d}p. \tag{A15}$$

Integrating by parts the second term of the r.h.s. of equation (A15) yields

$$A(h) = \mu_h + (-1)^{s-2} \left\{ \nu^{(s-2)}(p) GR^s(p) \Big|_0^{0.5} - \int_0^{0.5} \nu^{(s-1)} GR^s(p) \mathrm{d}p \right\}. \tag{A16}$$

Since by definition $GR^s(0) = 0$ and $\nu^{(s-2)}(0.5) = 0$ for all indices $A(h) \in \Omega_\rho$, the first term in the braces on the right hand side of the equation is nil. This yield

$$A(h) = \mu_h + (-1)^{s-1} \int_0^{0.5} \nu(s-1)(p) GR^s(p) \mathrm{d}p. \tag{A17}$$

Given that equations (A14) and (A17) both conform to the relation depicted in equation (A15), it follows that equation (A17) holds for all $s \in \{2, 3, \ldots\}$. Let $\Delta A_{12} = A(h_2) - A(h_1)$. From equation (A17), we get

$$\Delta A_{12} = \mu_{h2} = \mu_{h1} + (-1)^{s-1} \int_0^{0.5} \nu^{(s-1)}(p) \left[GR_2^s(p) - GR_1^s(p) \right] \mathrm{d}p. \tag{A18}$$

Note that $(-1)^{s-1} \nu^{(s-1)}(p)$ is non positive. This implies that if $GR_2^s(p) \geq GR_1^s(p)$ for all $p \in [0, 0.5]$, then $(-1)^{s-1} \int_0^{0.5} \nu^{(s-1)}(p) \left[GR_2^s(p) - GR_1^s(p) \right] \mathrm{d}p \geq 0$. If in addition, $\mu_{h2} \leq \mu_{h1}$, then $\Delta A_{12} \leq 0$. This proves for sufficiency of the condition.

Having provided a sufficiency condition let us now prove for the necessity of the condition. In order to prove necessity, we need to consider three cases:

1. $\mu_{h1} < \mu_{h2}$ together with $GR_2^s(p) \geq GR_1^s(p)$ for all $p \in [0, 0.5]$
2. $GR_2^s(p) < GR_1^s(p)$ on some arbitrary small interval $[p_c, p_c + \varepsilon]$ together with $\mu_{h1} = \mu_{h2}$
3. $GR_2^s(p) < GR_1^s(p)$ on some arbitrary small interval $[p_c, p_c + \varepsilon]$ together with $\mu_{h1} > \mu_{h2}$

Case 1: Consider the set of indices $A(h) \in \Omega_\rho^s$ for which $\nu^{(s-2)}(p)$ is constant for all $p \in [0, 0.5]$. This weight function $\nu(p)$ satisfies the conditions in the definition of Ω_ρ^s. Since

$\nu^{(s-1)}(p) = 0$ for all $p \in [0, 0.5]$, $(-1)^{s-1} \int_0^{0.5} \nu^{(s-1)}(p) \left[GR_2^s(p) - GR_1^s(p)\right] \mathrm{d}p = 0$. From equation (A18) this implies that $\Delta A_{12} > 0$. Hence it cannot be that $\mu_{h1} < \mu_{h2}$.

Case 2: Consider the set of indices $A(h) \in \Omega_\rho^s$ for which $\nu^{(s-2)}(p)$ takes the following form:

$$\nu^{(s-2)}(p) = \begin{cases} (-1)^{s-1}\varepsilon & 0 \le p_c \\ (-1)^{s-1}\left[p_c + \varepsilon - p\right] & p_c \le p \le p_c + \varepsilon \\ 0 & p \ge p_c + \varepsilon \end{cases} \tag{A19}$$

where $p_c \in [0, 0.5]$. Since $\nu(p)$ is differentiable almost everywhere, it satisfies the conditions in the definition of Ω_ρ^s. Differentiating equation (A19) yields

$$\nu^{(s-1)}(p) = \begin{cases} 0 & 0 \le p_c \\ (-1)^s & p_c \le p \le p_c + \varepsilon \\ 0 & p \ge p_c + \varepsilon \end{cases} \tag{A20}$$

Imagine now that $GR_2^s(p) < GR_1^s(p)$ on an interval $[p_c, p_c + \varepsilon]$ for ε that can be arbitrarily close to 0. For any $\nu(p)$ obeying the relation in equation (A19), the expression in equation (A18) is negative. Hence it cannot be that $GR_2^s(p) < GR_1^s(p)$ for $p \in [p_c, p_c+\varepsilon]$ if $\mu_{h1} = \mu_{h2}$.

Case 3: Consider the set of indices $A(h) \in \Omega_\rho^s$ for which $\nu^{(s-2)}(p)$ takes the following form:

$$\nu^{(s-2)}(p) = \begin{cases} (-1)^{s-1}\kappa & 0 \le p_c \\ (-1)^{s-1}\kappa\left[p_c + \varepsilon - p\right] & p_c \le p \le p_c + \varepsilon \\ 0 & p \ge p_c + \varepsilon \end{cases} \tag{A21}$$

where $\kappa > \left(\frac{\mu_{h1} - \mu_{h2}}{\varepsilon}\right)$ and $p_c \in [0, 0.5]$. Since $\nu(p)$ is differentiable almost everywhere, it satisfies the conditions in the definition of Ω_ρ^s. Differentiating equation (A21) yields

$$\nu^{(s-1)}(p) = \begin{cases} 0 & 0 \le p_c \\ (-1)^s\kappa & p_c \le p \le p_c + \varepsilon \\ 0 & p \ge p_c + \varepsilon \end{cases} \tag{A22}$$

Imagine now that $GR_2^s(p) < GR_1^s(p)$ on an interval $[p_c, p_c + \varepsilon]$ for ε that can be arbitrarily close to 0. For any $\nu(p)$ obeying the relation in equation (A21), the expression in equation (A19) is negative. Hence it cannot be that $GR_2^s(p) < GR_1^s(p)$ for $p \in [p_c, p_c+\varepsilon]$ if $\mu_{h1} > \mu_{h2}$.

Cases 1 to 3 prove the necessity of the condition. ■

A2 Construction of $C^s(p)$ and $R^s(p)$ estimators

A2.1 Estimator for $C^s(p)$

As seen earlier, the health concentration curve $C(p)$ is defined as follows

$$C(p) = \frac{1}{\mu_h} \int_0^p h(u)\mathrm{d}u. \tag{A23}$$

It can be re-written as

$$C(p) = \frac{1}{\mu_h} \int_0^1 \mathbb{1}(u < p) h(u)\mathrm{d}u. \tag{A24}$$

Apply the transformation $y = F_Y^{-1}(u)$ (with jacobian term $f_Y(y)$)

$$C(p) = \frac{1}{\mu_h} \int_0^\infty \mathbb{1}(y < F_Y^{-1}(p)) h(F_Y(y)) f_Y(y)\mathrm{d}y \tag{A25}$$

Let $f_{H|Y}$ be the conditional density of H on Y insert the following definition of the conditional expectation

$$E[H|Y = y] = \int_0^\infty h f_{H|Y}(h|y)\mathrm{d}h \tag{A26}$$

in equation A25 and using the definition for the joint density $f_{H,Y} = f_{H|Y} f_Y$, we get

$$C(p) = \frac{1}{\mu_h} \int_0^\infty \int_0^\infty h \mathbb{1}(y < F_Y^{-1}(p)) f_{H,Y}(h, y)\mathrm{d}h\mathrm{d}y, \tag{A27}$$

which gives the simple estimator for $C(p)$ from a sample (y_i, h_i) for $i = 1, \ldots, n$:

$$\hat{C}(p) = \frac{1}{N\bar{h}} \sum_{i=1}^{N} h_i \mathbb{1}(y_i \leqslant \hat{F}_Y^{-1}(p)). \tag{A28}$$

Here $\bar{h}$ is the sample average and $\hat{F}_Y^{-1}$ is a non-parametric estimator of the quantile function of Y based on the order statistics of (y_i).

Estimators for $C^s(p)$ could be recursively derived from that of $C(p)$ (derivation of this result is in section A2.3).

$$\hat{C}^s(p) = \frac{1}{N\bar{h}} \sum_{i=1}^{N} h_i \frac{(p - \hat{F}_Y(y_i))^{s-2}}{(s-1)!} \mathbb{1}(y_i \leqslant \hat{F}_Y^{-1}(p)) \tag{A29}$$

■

A2.2 Estimator for $R^s(p)$

In a similar fashion we can construct an estimator for R. Let us first rewrite $R(p)$ in the same form as $C(p)$:

$$\mu_h R(p) = \int_0^p r(u)\mathrm{d}u \tag{A30}$$

Given that Define $r(u)$ can be written as $h(1-u)-h(u)$ for $u \in [0,1]$, we can re-write this relationship as follows:

$$\mu_h R(p) = \int_{1-p}^1 h(u)\mathrm{d}u - \int_0^p h(u)\mathrm{d}u, \tag{A31}$$

which can be re-written as follows:

$$\mu_h R(p) = \int_0^1 [\mathbb{1}(u > 1-p) - \mathbb{1}(u < p)]h(u)\mathrm{d}u \tag{A32}$$

If ones defines a new variable $t = 1-u$, then $u = \phi(t) = 1-t$. In this framework,

$$\begin{aligned}
\int_0^p h(1-u)\mathrm{d}u &= \int_1^{1-p} h\left(1-\phi(t)\right)\phi'(t)\mathrm{d}t && \text{(A33)}\\
&= \int_1^{1-p} h\left(t\right)(-1)\mathrm{d}t && \text{(A34)}\\
&= \int_{1-p}^1 h(t)\mathrm{d}t && \text{(A35)}
\end{aligned}$$

The above sequence you have written should be

$$\begin{aligned}
\mu_h R(p) &= \int_0^p r(u)\mathrm{d}u && \text{(A36)}\\
&= \int_0^p h(1-u)\mathrm{d}u - \int_0^p h(u)\mathrm{d}u && \text{(A37)}\\
&= -\int_1^{1-p} h(u)\mathrm{d}u - \int_0^p h(u)\mathrm{d}u && \text{(A38)}\\
&= \int_{1-p}^1 h(u)\mathrm{d}u - \int_0^p h(u)\mathrm{d}u && \text{(A39)}
\end{aligned}$$

Furthermore, we could deduce that

$$\mu_h R(p) = \int_0^1 [\mathbb{1}(u > 1-p) - \mathbb{1}(u < p)]h(u)\mathrm{d}u \tag{A40}$$

This expression, upon applying a transformation $y = F_Y^{-1}(u)$, expanding the formula for h, yields

$$R(p) \times \mu_h = \int_0^\infty \int_0^\infty h[\mathbb{1}(y > F_Y^{-1}(1-p))] f_{H,Y}(h,y) \mathrm{d}h \mathrm{d}y \tag{A41}$$

$$- \int_0^\infty \int_0^\infty h[\mathbb{1}(y < F_Y^{-1}(p))] f_{H,Y}(h,y) \mathrm{d}h \mathrm{d}y \tag{A42}$$

which yields the estimator of $R(p)$

$$\hat{R}(p) = \frac{1}{N\bar{h}} \left\{ \sum_{i=1}^N h_i [\mathbb{1}(y_i > \hat{F}_Y^{-1}(1-p))] - \sum_{i=1}^N h_i [\mathbb{1}(y_i \leqslant \hat{F}_Y^{-1}(p))] \right\} \tag{A43}$$

As for C^s, it is possible to recursively compute estimators of R^s by first plugging the estimators of $\hat{R}$ and then by recursively computing (derivation of this result is in section A2.3)

$$\hat{R}^s(p) = \int_0^p \hat{R}^{s-1}(u) \mathrm{d}u, \tag{A44}$$

and

$$\begin{aligned} \widehat{R}^s(p) &= \frac{1}{N\bar{h}} \sum_{i=1}^N h_i \frac{1}{(s-1)!} p^{s-2} (p + (s-1)[\hat{F}_Y(y_i) - 1]) [\mathbb{1}(y_i > \hat{F}_Y^{-1}(1-p))] \\ &\quad - \frac{1}{N\bar{h}} \sum_{i=1}^N h_i \frac{(p - \hat{F}_Y(y_i))^{s-2}}{(s-1)!} [\mathbb{1}(y_i \leqslant \hat{F}_Y^{-1}(p))] \end{aligned} \tag{A45}$$

are the resulting estimators. ■

A2.3 Computation of integrals containing indicator variables involving inverse of $\hat{F}_Y$

Even though $\hat{F}_Y$ is a step function, the following standard result holds: $y_i \leqslant \hat{F}_Y^{-1}(p)$ if and only if $\hat{F}_Y(y_i) \leqslant p$. In what follows, We will check the formula for the estimator by induction.

First set $I_1(p) = \int_0^p \mathbb{1}(y_i \leqslant \hat{F}_Y^{-1}(u))\mathrm{d}u$ and compute

$$\begin{aligned} \int_0^p \mathbb{1}(y_i \leqslant \hat{F}_Y^{-1}(u))\mathrm{d}u &= \int_0^p \mathbb{1}(\hat{F}_Y(y_i) \leqslant u)\mathrm{d}u \quad \text{(A46)} \\ &= (p - \hat{F}_Y(y_i))\mathbb{1}(\hat{F}_Y(y_i) \leqslant p). \quad \text{(A47)} \end{aligned}$$

Then recursively compute

$$\begin{aligned} I_k(p) &= \int_0^p I_{k-1}(u)\mathrm{d}u \quad \text{(A48)} \\ &= \int_0^p \frac{(u - \hat{F}_Y(y_i))^{k-1}}{k!}\mathbb{1}(\hat{F}_Y(y_i) \leqslant u)\mathrm{d}u \quad \text{(A49)} \\ &= \frac{(p - \hat{F}_Y(y_i))^k}{(k+1)!}\mathbb{1}(\hat{F}_Y(y_i) \leqslant p). \quad \text{(A50)} \end{aligned}$$

By making the change of variable $s = k + 2$, the result follows. ■

In order to compute integrals containing indicator variables involving the (quantile) inverse of $\hat{F}_Y$, it is important to make a previous argument more explicit. In fact because $\hat{F}_Y$ is non-decreasing $\{y_i : \hat{F}_Y(y_i) \geqslant p\}$ is unbounded from above and because $\hat{F}_Y$ is right-continuous, $\{y_i : \hat{F}_Y(y_i) \geqslant p\}$ is closed to the left, thus it is closed at its infimum. However, by the definition of the quantile function,

$$\hat{F}_Y^{-1}(p) = \inf_{y_i}\{y_i : \hat{F}_Y(p) \geqslant p\}, \quad \text{(A51)}$$

we get the set equality

$$\{y_i : \hat{F}_Y(y_i) \geqslant p\} = [\hat{F}_Y^{-1}(p), \infty) \quad \text{(A52)}$$

This set inequality shows that $\hat{F}_Y^{-1}(p) \leqslant y_i$ if and only if $p \leqslant \hat{F}_Y(y_i)$. Taking complements of the set equality in $[0, \infty)$ yields the equality

$$\{y_i : \hat{F}_Y(y_i) < p\} = [0, \hat{F}_Y^{-1}(p))], \quad \text{(A53)}$$

which implies $y_i < \hat{F}_Y^{-1}(p)$ if and only if $\hat{F}_Y(y_i) < p$.This allows us to compute the following integral,

$$\begin{aligned}
\int_0^p \mathbb{1}(y_i > \hat{F}_Y^{-1}(1-u))\mathrm{d}u &= \int_0^p \mathbb{1}(\hat{F}_Y(y_i) > (1-u))\mathrm{d}u && \text{(A54)}\\
&= \int_{1-p}^1 \mathbb{1}(\hat{F}_Y(y_i) > u)\mathrm{d}u && \text{(A55)}\\
&= \mathbb{1}(\hat{F}_Y(y_i) > (1-p))(\hat{F}_Y(y_i) - 1 + p). && \text{(A56)}
\end{aligned}$$

From equation (A57), it is clear that integrating recursively, we should obtain at step k an integrand of the form

$$\frac{1}{k!}p^{k-1}(p + k[\hat{F}_Y(y_i) - 1])\mathbb{1}(\hat{F}_Y(y_i) > (1-p)), \tag{A57}$$

resulting at step $k+1$ in an integrand of the form

$$\frac{1}{(k+1)!}p^{k}(p + (k+1)[\hat{F}_Y(y_i) - 1])\mathbb{1}(\hat{F}_Y(y_i) > (1-p)). \tag{A58}$$

We could verify that by induction. Since we checked for $k = 1$, what remains to do is to check for an arbitrary k and see if we get the correct form for $k+1$.

Set $J_1(p) = \mathbb{1}(y_i > \hat{F}_Y^{-1}(1-u))\mathrm{d}u$ and recursively compute

$$\begin{aligned}
J_k(p) &= \int_0^p J_{k-1}(u)\mathrm{d}u && \text{(A59)}\\
&= \int_0^p \mathbb{1}(\hat{F}_Y(y_i) > (1-u))\frac{1}{k!}u^{k-1}(u + k[\hat{F}_Y(y_i) - 1])\mathrm{d}u && \text{(A60)}\\
&= \mathbb{1}(\hat{F}_Y(y_i) > (1-p))\frac{1}{k!}\int_0^p u^{k-1}(u + k[\hat{F}_Y(y_i) - 1])\mathrm{d}u && \text{(A61)}\\
&= \mathbb{1}(\hat{F}_Y(y_i) > (1-p))\frac{1}{k!}\left[\frac{p^{k+1}}{k+1} + \frac{p^k}{k}k[\hat{F}_Y(y_i) - 1]\right] && \text{(A62)}\\
&= \mathbb{1}(\hat{F}_Y(y_i) > (1-p))\frac{p^k}{(k+1)!}[p + (k+1)[\hat{F}_Y(y_i) - 1]] && \text{(A63)}
\end{aligned}$$

By making the change of variable $s = k + 2$, the result follows. ∎

A3 Bootstrap procedure

As suggest by Linton et al. (2005) and Shechtman et al. (2008), we used a recentered bootstrap procedure. The bootstrap algorithm for B repetitions is constructed as follows:

1. Repeat for $b = 1, \ldots, B$

    - Draw a sample of size n_1 from $\mathcal{S}_1$. Compute the nonparametric estimator $\hat{L}_{1b}$.
    - Draw a sample of size n_2 from $\mathcal{S}_2$. Compute the nonparametric estimator $\hat{L}_{2b}$.
    - Compute $\hat{L}_{12b}(p) = \hat{L}_{1b}(p) - \hat{L}_{2b}(p)$.
    - Compute $\hat{\tau}_b = \sup_p \sqrt{\frac{n_1 n_2}{n_1+n_2}}[\hat{L}_{12b}(p) - \hat{L}_{12}(p)]$.

2. Using the sample $\hat{\tau}_1, \ldots, \hat{\tau}_B$, compute the bootstrap p-value

$$\frac{1}{B}\sum_{b=1}^{B} \mathbb{1}(\hat{\tau}_b > \hat{\tau}).$$